$$-\begin{vmatrix} a_{11} & a_{12} & A_{13} & A_{14} \\ a_{12} & a_{22} & A_{23} & A_{24} \\ a_{13} & a_{23} & 0 & A_{34} \\ a_{14} & a_{24} & -A_{34} & 0 \end{vmatrix} - \begin{vmatrix} a_{11} & A_{12} & A_{13} & a_{14} \\ a_{12} & 0 & A_{23} & a_{24} \\ a_{13} & -A_{23} & 0 & a_{34} \\ a_{14} & -A_{24} & -A_{34} & a_{44} \end{vmatrix} - \begin{vmatrix} a_{11} & A_{12} & a_{13} & A_{14} \\ a_{12} & 0 & a_{23} & A_{24} \\ a_{13} & -A_{23} & a_{33} & A_{34} \\ a_{14} & -A_{24} & a_{34} & 0 \end{vmatrix}$$

$$-\begin{vmatrix} 0 & a_{12} & a_{13} & A_{14} \\ -A_{12} & a_{22} & a_{23} & A_{24} \\ -A_{13} & a_{23} & a_{33} & A_{34} \\ -A_{14} & a_{24} & a_{34} & 0 \end{vmatrix} - \begin{vmatrix} 0 & a_{12} & A_{13} & a_{14} \\ -A_{12} & a_{22} & A_{23} & a_{24} \\ -A_{13} & a_{23} & 0 & a_{34} \\ -A_{14} & a_{24} & -A_{34} & a_{44} \end{vmatrix} - \begin{vmatrix} 0 & A_{12} & a_{13} & a_{14} \\ -A_{12} & 0 & a_{23} & a_{24} \\ -A_{13} & -A_{23} & a_{33} & a_{34} \\ -A_{14} & -A_{24} & a_{34} & a_{44} \end{vmatrix}$$

To the first of the six we apply Laplace's expansion, the dividing line separating the first two *columns* from the last two. For the other five, we first interchange columns to bring the a_{ij} into the first and second columns and then apply Laplace's expansion. In their final sum, the coefficient of $-A_{12}A_{34}$ is

$$\begin{vmatrix} a_{12} & a_{14} \\ a_{23} & a_{34} \end{vmatrix} - \begin{vmatrix} a_{12} & a_{13} \\ a_{24} & a_{34} \end{vmatrix} + \begin{vmatrix} a_{12} & a_{23} \\ a_{14} & a_{34} \end{vmatrix} - \begin{vmatrix} a_{12} & a_{24} \\ a_{13} & a_{34} \end{vmatrix} = 2 \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix}$$

Similar simplifications may be made in the coefficients of $A_{13}A_{24}$ and $A_{14}A_{23}$. A term A_{ij}^2 occurs singly. A term $A_{ij}A_{kl}$ in which two and but two of the subscripts i, j, k, l are equal occurs twice, with equal coefficients. We obtain the final result $D=a+A-S$, where (denoting a_{ij} by ij)

$$S=A_{12}^2 \begin{vmatrix} 33 & 34 \\ 34 & 44 \end{vmatrix} + A_{13}^2 \begin{vmatrix} 22 & 24 \\ 24 & 44 \end{vmatrix} + A_{14}^2 \begin{vmatrix} 22 & 23 \\ 23 & 33 \end{vmatrix} + A_{23}^2 \begin{vmatrix} 11 & 14 \\ 14 & 44 \end{vmatrix}$$

$$+A_{24}^2 \begin{vmatrix} 11 & 13 \\ 13 & 33 \end{vmatrix} + A_{34}^2 \begin{vmatrix} 11 & 12 \\ 12 & 22 \end{vmatrix} + 2A_{12}A_{34} \begin{vmatrix} 13 & 14 \\ 23 & 24 \end{vmatrix} + 2A_{13}A_{24} \begin{vmatrix} 12 & 14 \\ 23 & 34 \end{vmatrix}$$

$$+2A_{14}A_{23} \begin{vmatrix} 12 & 13 \\ 24 & 34 \end{vmatrix} - 2A_{12}A_{13} \begin{vmatrix} 23 & 24 \\ 34 & 44 \end{vmatrix} + 2A_{12}A_{14} \begin{vmatrix} 23 & 24 \\ 33 & 34 \end{vmatrix}$$

$$-2A_{12}A_{24} \begin{vmatrix} 13 & 14 \\ 33 & 34 \end{vmatrix} - 2A_{13}A_{14} \begin{vmatrix} 22 & 24 \\ 23 & 34 \end{vmatrix} - 2A_{13}A_{23} \begin{vmatrix} 12 & 14 \\ 24 & 44 \end{vmatrix}$$

$$-2A_{13}A_{34} \begin{vmatrix} 12 & 14 \\ 22 & 24 \end{vmatrix} - 2A_{14}A_{24} \begin{vmatrix} 12 & 13 \\ 23 & 33 \end{vmatrix} + 2A_{14}A_{34} \begin{vmatrix} 12 & 13 \\ 22 & 23 \end{vmatrix}$$

$$-2A_{23}A_{24} \begin{vmatrix} 11 & 13 \\ 14 & 34 \end{vmatrix} - 2A_{24}A_{34} \begin{vmatrix} 11 & 13 \\ 12 & 23 \end{vmatrix} + 2A_{12}A_{23} \begin{vmatrix} 13 & 14 \\ 34 & 44 \end{vmatrix}$$

$$+2A_{23}A_{34} \begin{vmatrix} 11 & 14 \\ 12 & 24 \end{vmatrix}.$$

A law for the simplified expansion of D, which may be readily extended to like determinants of any order, may now be given. The law may be expressed very luminously by use of a "multiplication table." As in the usual notation

for quadratic forms, a term $2cxy$ is conveniently expressed by $cxy+cyx$. Then S is given by the accompanying multiplication table.

	A_{34}	A_{24}	A_{23}	A_{14}	A_{13}	A_{12}
A_{34}	$\begin{vmatrix} 11 & 12 \\ 12 & 22 \end{vmatrix}$	$-\begin{vmatrix} 11 & 13 \\ 12 & 23 \end{vmatrix}$	$\begin{vmatrix} 11 & 14 \\ 12 & 24 \end{vmatrix}$	$\begin{vmatrix} 12 & 13 \\ 22 & 25 \end{vmatrix}$	$-\begin{vmatrix} 12 & 14 \\ 22 & 24 \end{vmatrix}$	$\begin{vmatrix} 13 & 14 \\ 23 & 24 \end{vmatrix}$
A_{24}	$-\begin{vmatrix} 11 & 13 \\ 12 & 23 \end{vmatrix}$	$\begin{vmatrix} 11 & 13 \\ 13 & 33 \end{vmatrix}$	$-\begin{vmatrix} 11 & 14 \\ 13 & 34 \end{vmatrix}$	$-\begin{vmatrix} 12 & 13 \\ 23 & 33 \end{vmatrix}$	$\begin{vmatrix} 12 & 14 \\ 23 & 34 \end{vmatrix}$	$-\begin{vmatrix} 13 & 14 \\ 33 & 34 \end{vmatrix}$
A_{23}	$\begin{vmatrix} 11 & 12 \\ 14 & 24 \end{vmatrix}$	$-\begin{vmatrix} 11 & 13 \\ 14 & 34 \end{vmatrix}$	$\begin{vmatrix} 11 & 14 \\ 14 & 44 \end{vmatrix}$	$\begin{vmatrix} 12 & 13 \\ 24 & 34 \end{vmatrix}$	$-\begin{vmatrix} 12 & 14 \\ 24 & 44 \end{vmatrix}$	$\begin{vmatrix} 13 & 14 \\ 34 & 44 \end{vmatrix}$
A_{14}	$-\begin{vmatrix} 12 & 22 \\ 13 & 23 \end{vmatrix}$	$-\begin{vmatrix} 12 & 23 \\ 13 & 33 \end{vmatrix}$	$\begin{vmatrix} 12 & 24 \\ 13 & 34 \end{vmatrix}$	$\begin{vmatrix} 22 & 23 \\ 23 & 33 \end{vmatrix}$	$-\begin{vmatrix} 22 & 24 \\ 23 & 34 \end{vmatrix}$	$\begin{vmatrix} 23 & 24 \\ 33 & 34 \end{vmatrix}$
A_{13}	$-\begin{vmatrix} 12 & 22 \\ 14 & 24 \end{vmatrix}$	$\begin{vmatrix} 12 & 23 \\ 14 & 34 \end{vmatrix}$	$-\begin{vmatrix} 12 & 24 \\ 14 & 44 \end{vmatrix}$	$-\begin{vmatrix} 22 & 23 \\ 24 & 34 \end{vmatrix}$	$\begin{vmatrix} 22 & 24 \\ 34 & 44 \end{vmatrix}$	$-\begin{vmatrix} 23 & 24 \\ 34 & 44 \end{vmatrix}$
A_{12}	$\begin{vmatrix} 13 & 23 \\ 14 & 24 \end{vmatrix}$	$-\begin{vmatrix} 13 & 33 \\ 14 & 34 \end{vmatrix}$	$\begin{vmatrix} 13 & 34 \\ 14 & 44 \end{vmatrix}$	$\begin{vmatrix} 23 & 33 \\ 24 & 34 \end{vmatrix}$	$-\begin{vmatrix} 23 & 34 \\ 24 & 44 \end{vmatrix}$	$\begin{vmatrix} 33 & 34 \\ 34 & 44 \end{vmatrix}$

The law of formation is now evident. The body of the table gives in proper position and with proper sign the minors of $|\ a_{ij}\ | \equiv a$. It follows (Muir, *Theory of Determinants*, §174) that the determinant of the sixth order defined by the table equals a^3, being the second compound of a.

4. Changing the notation by replacing A_{13} and A_{24} by their negatives, we obtain from S a quadratic form S' with the signs of its terms all positive, while A remains unaltered. The body of the multiplication table is altered only in having all its signs made positive. If $a \neq 0$, the resulting matrix M defines a linear transformation on the six variables A_{ij}. It is known (author's *Linear Groups*, pp. 145-155) to leave invariant the quadratic function

$$F \equiv A_{12}A_{34} - A_{13}A_{24} + A_{14}A_{23},$$

whose square equals A (§3). Of several geometrical interpretations, the most elegant consists in regarding the A_{ij} as the six line-coödinates in ordinary space, so that $F \equiv 0$. The matrix M therefore defines a transformation of the straight lines in space. We may also interpret the equation $D=a$ (or $S=0$) as the totality of straight lines whose six line-coördinates are subject to a quadratic condition.

NOTE ON THE NECESSARY CONDITION THAT TWO LINEAR HOMOGENEOUS DIFFERENTIAL EQUATIONS SHALL HAVE COMMON INTEGRALS.*

By IDA MAY SCHOTTENFELS.

Professor Von Escherich in the *Denkschriften der Wiener Akademie*, Vol. 46, and later Heffter in Crelle's *Journal*, Vol. 116, proved that there exists for linear differential homogeneous equations a theory analogous to that of algebraic equations, confining their researches to the analogues to theorems upon the Highest Common Factor and Lowest Common Multiple.

During the past year Dr. Epsteen and Dr. Pierce revived this subject, and in THE AMERICAN MATHEMATICAL MONTHLY, Vol. X, March, 1903, pp. 63-68, Dr. Epsteen gives the necessary condition for one common integral, while Dr. Pierce gives the sufficient condition for $k \gtreqless 1$ independent common integrals.

This note gives the necessary condition for two common integrals.

The following example illustrates the method.

$$a_0x^4+a_1x^3+a_2x^2+a_3x+a_4=0, \tag{1}$$

$$b_0x^3+b_1x^2+b_2x+b_3=0. \tag{2}$$

To find the necessary condition that equations (1) and (2) have two roots in common. According to the well known dialytic method of elimination of Sylvester we may write

$$\left\{\begin{array}{rrrrrr} a_0x^5+a_1x^4+a_2x^3+a_3x^2+a_4x & & & & & =0, \\ & a_0x^4+a_1x^3+a_2x^2+a_3x+a_4 & & & & =0, \\ b_0x^5+b_1x^4 & +b_2x^3+b_3x^2 & & & & =0, \\ & b_0x^4 & +b_1x^3+b_2x^2+b_3x & & & =0, \\ & & b_0x^3+b_1x^2+b_2x+b_3 & & & =0. \end{array}\right. \tag{3}$$

Eliminating x^5, x^4, x^3, x^2, from (3) we get

$$\begin{vmatrix} a_0 & a_1 & a_2 & a_3 & a_4x+0 \\ 0 & a_0 & a_1 & a_2 & a_3x+a_4 \\ b_0 & b_1 & b_2 & b_3 & 0x+0 \\ 0 & b_0 & b_1 & b_2 & b_3x+0 \\ 0 & 0 & 0 & b_1 & b_2x+b_3 \end{vmatrix} =0. \tag{4}$$

Since (4) shall hold for at least two values of x, and yet is linear in x, it is an identity in x, and hence

* Presented to the American Mathematical Society (New York) October 30, 1903.

$$(5)\qquad \begin{vmatrix} a_0 & a_1 & a_2 & a_3 & a_4 \\ 0 & a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 & 0 \\ 0 & b_0 & b_1 & b_2 & b_3 \\ 0 & 0 & 0 & b_1 & b_2 \end{vmatrix} \equiv 0, \qquad \begin{vmatrix} a_0 & a_1 & a_2 & a_3 & 0 \\ 0 & a_0 & a_1 & a_2 & a_4 \\ b_0 & b_1 & b_2 & b_3 & 0 \\ 0 & b_0 & b_1 & b_2 & 0 \\ 0 & 0 & 0 & b_1 & b_3 \end{vmatrix} \equiv 0.$$

Hence the required necessary condition that equations (1) and (2) have two roots in common is the vanishing of determinants (5).

The generalizatisn to equations of degree m aud n, that have p roots in common follows readily. The equation of degree m must be multiplied by x^{n-p} while the one of degree n must be multiplied by x^{m-p}.

In exactly the same manner the analog of this method for linear homogeneous differential equations can be developed. The theorem that a linear homogeneous differential equation of the nth order has n and only n linearly independent integrals is employed, and the necessary condition is expressed in linear homogeneous differential operators, and has a form analogous to that of the necessary condition (5).

Given the following two linear homogeneous differential equations:

$$(\mathrm{I})\qquad a_0(x)y^{\mathrm{iv}}+a_1(x)y'''+a_2(x)y''+a_3(x)y'+a_4(x)y=0,$$

$$(\mathrm{II})\qquad \beta_0(x)y'''+\beta_1(x)y''+\beta_2(x)y'+\beta_3(x)y=0.$$

If these two equations I and II have at least one common integral, we can by differentiation write,

$$(\mathrm{III})\begin{cases} a_0y^{\mathrm{v}}+(a_0'+a_1)y^{\mathrm{iv}}+(a_1'+a_2)y'''+(a_2'+a_3)y''+(a_3'+a_4)y'+a_4'.y\equiv 0 \\ \qquad a_0y^{\mathrm{iv}}+\qquad a_1y'''+\qquad a_2y''+\qquad a_3y'+a_4.y\equiv 0 \\ \beta_0y^{\mathrm{v}}+(2\beta_0'+\beta_1)y^{\mathrm{iv}}+(\beta_0''+2\beta_1'+\beta_2)y'''+(\beta_1''+2\beta_2'+\beta_3)y'' \\ \qquad\qquad +(\beta_2''+2\beta_3')y'+\beta_3''.y\equiv 0 \\ \qquad \beta_0y^{\mathrm{iv}}+(\beta_0'+\beta_1)y'''+(\beta_1'+\beta_2)y''+(\beta_2'+\beta_3)y'+\beta_3'.y\equiv 0 \\ \qquad\qquad \beta_0y'''+\qquad \beta_1y''+\qquad \beta_2y'+\beta_3.y\equiv 0. \end{cases}$$

Eliminating y^{v}, y^{iv}, y''', y'' from (III) we get

$$(\mathrm{IV})\qquad \begin{vmatrix} a_0 & a_0'+a_1 & a_1'+a_2 & a_2'+a_3 & (a_3'+a_4)y'+a_4'.y \\ 0 & a_0 & a_1 & a_2 & a_3y'+a_4.y \\ \beta_0 & 2\beta_0'+\beta_1 & \beta_0''+2\beta_1'+\beta_2 & \beta_1''+2\beta_2'+\beta_3 & (\beta_2''+2\beta_3')y'+\beta_3''.y \\ 0 & \beta_0 & \beta_0'+\beta_1 & \beta_1'+\beta_2 & (\beta_2'+\beta_3)y'+\beta_3'.y \\ 0 & 0 & \beta_0 & \beta_2 & \beta_2y'+\beta_3y \end{vmatrix} \equiv 0$$

which may be written $\lambda y'+\mu y=0$.

If there are two independent common integrals, IV must hold for two linearly independent functions y (one not a constant times the other), whence $\lambda=0$, $\mu=0$, or

$$(V_1)\quad \begin{vmatrix} a_0 & a_0'+a_1 & a_1'+a_2 & a_2'+a_3 & a_3'+a_4 \\ 0 & a_0 & a_1 & a_2 & a_3 \\ \beta_0 & 2\beta_0'+\beta_1 & \beta_0''+2\beta_1'+\beta_2 & \beta_1''+2\beta_2'+\beta_3 & \beta_2''+2\beta_3' \\ 0 & \beta_0 & \beta_0'+\beta_1 & \beta_1'+\beta_2 & \beta_2'+\beta_3 \\ 0 & 0 & \beta_0 & \beta_1 & \beta_2 \end{vmatrix} \equiv 0,$$

$$(V_2)\quad \begin{vmatrix} a_0 & a_0'+a_1 & a_1'+a_2 & a_2'+a_3 & a_4' \\ 0 & a_0 & a_1 & a_2 & a_4 \\ \beta_0 & 2\beta_0'+\beta_1 & \beta_0''+2\beta_1'+\beta_2 & \beta_1''+2\beta_2'+\beta_3 & \beta_3'' \\ 0 & \beta_0 & \beta_0'+\beta_1 & \beta_1'+\beta_2 & \beta_3' \\ 0 & 0 & \beta_0 & \beta_1 & \beta_3 \end{vmatrix} \equiv 0.$$

Hence V_1 and V_2 furnish the required necessary condition.

NEW YORK CITY, *September, 1903.*

LINEAR COVARIANTS OF THE BINARY QUADRATIC AND CUBIC.

By L. C. WALKER, Professor of Mathematics, Colorado School of Mines, Golden, Col.

The definition of *weight* is that every coefficient is of weight w measured by its suffix, and that every product of coefficients is of weight measured by the sum of the suffixes of its various factors.

A semi-covariant of the two quantics is a function of the two sets of coefficients, which is homogeneous in each set separately, and *isobaric* (equal weight) on the whole, though not necessarily in the sets separately.

The practice of speaking of a covariant whose dimensions are partial degrees i_1, i_2 in the two sets of coefficients and ω in the variables has of late become almost universal.*

The degrees of quantics in the variables are generally† spoken of as their orders p_1, p_2.

The order ω, the partial degrees i_1, i_2 in the coefficients of the binary quadratic and cubic

$$(a_0, a_1, a_2)(x, y)^2, \quad (b_0, b_1, b_2, b_3)(x, y)^3,$$

and the weight w of the semi-invariant which is the leading coefficient C_0 in the linear covariant, are connected by the relation $i_1p_1+i_2p_2-\omega=2w$.

Here $p_1=2$, $p_2=3$, $\omega=1$, whence $2i_1+3i_2-1=2w$. More generally, if m be any positive integer and n any positive odd integer, we have, from the conditions of linear covariancy, $2mi_1+3ni_2-1=2w$. Thus the binary quadratic and cubic have an indefinite number of linear covariants.

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*Elliott's *Algebra of Quantics*. †*Ibid.*

We now shall find the linear covariants,

$$(a)\quad (2;\,1,\,2;\,1,\,3); \qquad (b)\quad (3;\,2,\,2;\,1,\,3);$$
$$(c)\quad (5;\,1,\,2;\,3,\,3); \qquad (d)\quad (6;\,2,\,2;\,3,\,3).$$

(a). Assume for the semi-invariant the most general form

$$S\equiv C_0\equiv a_2b_2+\lambda_1a_1b_1+\lambda_2a_2b_0,$$

where λ_1, λ_2 are arbitrary multipliers. Operate on this with the two annihilators

$$\Omega_1\equiv a_0\frac{d}{da_1}+2a_1\frac{d}{da_2}, \quad \Omega_2\equiv b_0\frac{d}{db_1}+2b_1\frac{d}{db_2}+3b_2\frac{d}{db_3},$$

and we obtain

$$a_0b_1[\lambda_1+2]+a_1b_0[2\lambda_2+\lambda_1];$$

for which to vanish we must have

$$\lambda_1+2=0, \quad 2\lambda_2+\lambda_1=0; \quad i.\,e.,\ \lambda_1=-2\lambda_2=-2.$$

$$\therefore C_0\equiv a_0b_2-2a_1b_1+a_2b_0,$$
$$C_1\equiv O_1C_0+O_2C_0\equiv a_0b_3-2a_1b_2+a_2b_1,$$

where we have used the annihilators

$$O_1\equiv 2a_1\frac{d}{da_0}+a_2\frac{d}{da_1}, \quad O_2\equiv 3b_1\frac{d}{db_0}+2b_2\frac{d}{db_1}+b_3\frac{d}{db_2}.$$

Thus the linear covariant is

I. $$(a_0b_2-2a_1b_1+a_2b_0)x+(a_0b_3-2a_1b_2+a_2b_1)y.$$

The second transvectant of the quadratic and cubic gives I.

(b). Including all possible terms, the semi-invariant is of the form

$$S\equiv C_0\equiv a_0^2b_3+\lambda_1a_0a_1b_2+\lambda_2a_0a_2b_1+\lambda_3a_1^2b_1+\lambda_4a_1b_2b_0.$$

Operate on this with Ω_1, Ω_2. The vanishing of the expression requires four linear equations in the λ's, from which we find

$$\lambda_1=-3, \quad \lambda_2=1, \quad \lambda_3=2, \quad \lambda_4=-1.$$

$$\therefore S\equiv C_0\equiv a_0^2b_3-3a_0a_1b_2+a_0a_2b_1+2a_1^2b_1-a_1a_2b_0,$$
$$C_1\equiv O_1C_0+O_2C_0\equiv -(a_2^2b_0-3a_1a_2b_1+a_0a_2b_2+2a_1^2b_2-a_0a_1b_3).$$

The linear covariant is

II. $$C_0x+C_1y.$$

The first transvectant of I and the quadratic gives II. The third transvectant of the cubic and the square of the quadratic gives II.

(*c*). Assume the semi-invariant to be

$$S\equiv C_0\equiv a_0b_0b_2b_3+\lambda_1a_0b_1^2b_3+\lambda_2a_0b_1b_2^2+\lambda_3a_1b_0b_1b_3 +\lambda_4a_1b_0b_2^3+\lambda_5a_1b_1^2b_2+\lambda_6a_2b_0^2b_3+\lambda_7a_2b_0b_1b_2+\lambda_8a_2b_1^3.$$

Solving as in (*a*), we obtain eight linear equations in the λ's, from which we find

$$\lambda_1=-2,\ \lambda_2=1,\ \lambda_3=2,\ \lambda_4=-4,\ \lambda_5=2,\ \lambda_6=-1,\ \lambda_7=3,\ \lambda_8=-2.$$

Thus the linear covariant is

III. $$\begin{aligned}&(a_0b_0b_2b_3-2a_0b_1^2b_3+a_0b_1b_2^2+2a_1b_0b_1b_3-4a_1b_0b_2^2\\&+2a_1b_1^2b_2-a_2b_0^2b_3+3a_2b_0b_1b_2-2a_2b_1^3)x\\&-(a_2b_0b_1b_3-2a_2b_0b_2^2+a_2b_1^2b_2+2a_1b_0b_2b_3-4a_1b_1^2b_3\\&+2a_1b_1b_2^2-a_0b_0b_3^2+3a_0b_1b_2b_3-2a_0b_2^3)y.\end{aligned}$$

The first transvectant of I and the *Hessian* of the cubic gives III. The second transvectant of the quadratic and the *cubicovariant* of the cubic gives III.

(*d*). Here we assume the semi-invariant to be

$$\begin{aligned}S\equiv C_0\equiv{}&a_0^2b_0b_3^2+\lambda_1a_0^2b_2^3+\lambda_2a_0^2b_1b_2b_3+\lambda_3a_0a_1b_0b_2b_3+\lambda_4a_0a_1b_1^2b_3\\&+\lambda_5a_0a_1b_1b_2^2+\lambda_6a_0a_2b_0b_1b_3+\lambda_7a_0a_2b_0b_2^2+\lambda_8a_0a_2b_1^2b_2\\&+\lambda_9a_1^2b_0b_1b_3+\lambda_{10}a_1^2b_0b_2^2+\lambda_{11}a_1^2b_1^2b_2+\lambda_{12}a_1a_2b_0^2b_3\\&+\lambda_{13}a_1a_2b_0b_1b_2+\lambda_{14}a_1a_2b_1^3+\lambda_{15}a_2^2b_0b_2+\lambda_{16}a_2^2b_0b_1^2,\end{aligned}$$

which includes all possible terms. As in (*a*), we obtain

$$\begin{aligned}&a_0^2b_1^2b_3(\lambda_4+2\lambda_1)+a_0a_2b_0^2b_3(\lambda_{12}+\lambda_6)+a_0a_2b_1^3(\lambda_{14}+2\lambda_8)+a_1^2b_0^2b_3(2\lambda_{12}+\lambda_9)\\&+a_1^2b_1^3(2\lambda_{14}+2\lambda_{11})+a_2^2b_0^2b_1(4\lambda_{16}+4\lambda_{15})+a_0^2b_0b_2b_3(\lambda_3+\lambda_1+6)+a_0^2b_1b_2^2\\&(\lambda_5+6\lambda_2+3\lambda_1)+a_1a_2b_0^2b_2(4\lambda_{15}+\lambda_{12}+3\lambda_{12})+a_1a_2b_0b_1^2(4\lambda_{16}+3\lambda_{14}+2\lambda_{13})+\\&a_0a_1b_0b_1b_3(2\lambda_9+2\lambda_6+2\lambda_4+2\lambda_3)+a_0a_1b_0b_2^2(2\lambda_{10}+2\lambda_7+\lambda_5+3\lambda_3)+a_0a_1b_1^2b_2\\&(2\lambda_{11}+2\lambda_8+4\lambda_5+3\lambda_4)+a_0a_2b_0b_1b_2(\lambda_{13}+2\lambda_8+4\lambda_7+3\lambda_6)+a_1^2b_0b_1b_2(2\lambda_{13}+2\lambda_{11}\\&+4\lambda_{10}+3\lambda_9).\end{aligned}$$

Its vanishing gives (5; 2, 2; 3, 3) relations which have to be satisfied by the (6; 2, 2; 3, 3) multipliers. If then (6; 2, 2; 3, 3) exceeds (5; 2, 2; 3, 3) we

can satisfy them, the number of the multipliers still arbitrary being $(6; 2, 2; 3, 3)-(5; 2, 2; 3, 3)\equiv 17-15=2$.

First, the first transvectant (1), of the cubic and the square of I; and (2), of the *Hessian* of the cubic and II: or, *second*, the second transvectant (1), of the cubic and the square of I; (2), of the quadratic and the product of I by the *Hessian* of the cubic; and (3), of the *Hessian* of the cubic and the product of the quadratic by I: or, *third*, the third transvectant of the *cubicovariant* of the cubic and the square of the quadratic—either *first* or *second* or *third* shows that the partitions* $a_2^2b_0b_2$ and $a_2^2b_0b_1^2$ are absent from this linear covariant. We then have $\lambda_{16}=\lambda_{15}=0$. Now from the other fourteen linear equations in the λ's we obtain

$$\lambda_1=2,\quad \lambda_2=\lambda_3=-3,\quad \lambda_4=6,\quad \lambda_5=-3,\quad \lambda_6=-1,\quad \lambda_7=2,$$
$$\lambda_8=-1,\quad \lambda_9=-2,\quad \lambda_{10}=4,\quad \lambda_{11}=-2,\quad \lambda_{12}=1,\quad \lambda_{13}=-3,\quad \lambda_{14}=2.$$

The required linear covariant is

$$(a_0^2b_0b_3-3a_0^2b_1b_2b_3+2a_0^2b_2^3-3a_0a_1b_0b_2b_3+6a_0a_1b_1^2b_3-3a_0a_1b_1b_2^2$$
$$-a_0a_2b_0b_1b_3+a_0a_2b_0b_2^2-a_0a_2b_1^2b_2-2a_1^2b_0b_1b_3+4a_1^2b_0b_2^2-2a_1^2b_1^2b_2$$
$$+a_1a_2b_0^2b_3-3a_1a_2b_0b_1b_2+2a_1a_2b_1^3)x+(a_0a_1b_0b_3^2-3a_0a_1b_1b_2b_3+2a_0a_1b_2^3$$
$$-a_0a_2b_0b_2b_3+2a_0a_2b_1^2b_2-a_0a_2b_1b_2^2-2a_1^2b_0b_2b_3+4a_1^2b_1^2b_3-2a_1^2b_1b_2^2$$
$$-3a_1a_2b_0b_1b_3+6a_1a_2b_0b_1^2-3a_1a_2b_1^2b_2+a_2^2b_0^2b_3-3a_2^2b_0b_1b_2+2a_2^2b_1^3)y.$$

The number of linearly independent semi-invariants of given weight w and partial degrees i_1, i_2 of the quadratic and cubic, is given by

$$(w; i_1, p_1; i_2, p_2)-(w-1; i_1, p_1; i_2, p_2).$$

This expression is not applicable when it exceeds unity for all values of m and n that do not give linear semi-invariants, because to each of these values corresponds one and only one linear covariant for a transvection of some combination (1), of the two quantics; or (2), of their covariants; or (3), of the quantics and their covariants. Professor Paul Gordan has proved that a complete system of transvectants is coextensive with a complete system of covariants, also including invariants as a particular case. For the geometrical interpretation of this system of quantics, see Art. 198 of Salmon's *Higher Algebra*.

* For the theory of numbers of partitions, see Professor Cayley's second memoir on Quantics (*Collected Works*, Vol. II).

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

169. Proposed F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

At what rate per cent. must a note be discounted at the end of a quarter of a year in order to produce a discount equivalent to 10% interest for the year?

No solution received.

170. Proposed by J. F. LAWRENCE, A. B., Breckenridge, Mo.

Suppose the market value of 5% bank stock to be 11 1-9% higher than 8% corporation bonds; I realize 8% on my investment, and my income from each is \$180. What did I invest in each?

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, West Va.

$11\frac{1}{9}\% = \frac{1}{9}$. $\therefore$ The 5% stock costs $\frac{1}{9}$ as much per share as the 8% stock.

$\therefore \dfrac{\text{Idvestment in 5\% stock}}{\frac{1}{9}} \times \frac{5}{100} = \text{Investment in 8\% stock} \div \frac{8}{100}$.

$\therefore$ Investment in 5% stock : Investment in 8% stock = 16 : 9.

Total investment = \$360 ÷ .08 = \$4500.

Investment in 5% stock = $\frac{16}{25}$ of \$4500 = \$2880.

Investment in 8% stock = $\frac{9}{25}$ of \$4500 = \$1620.

Also solved by *H. B. LEONARD, B. S.*, and *G. W. GREENWOOD, B. A.*

171. Proposed by JOHN S. ROYER, Editor of The School Visitor, Columbus, Ohio.

A drawer made of inch boards is 8 inches wide, 6 inches deep, and slides horizontally. How far must it be drawn out to put into it a book 4 inches thick, 6 inches wide, and 9 inches long?

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va., and H. B. LEONARD, B

Let a=length of book, b=its thickness, c=depth of drawer, x=distance it must be drawn out. If we draw out the drawer just far enough to put in the book lengthwise we have from the figure properly drawn, $a : x = c : b$ or $9 : x = 5 : 4$. Therefore $x = 7\frac{1}{5}$ inches.

ALGEBRA.

183. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

Find the condition that $x : y : z$ may be real, given that $\Sigma ax^2 = \Sigma axy = 0$ and $\Sigma a^2 = 1$.

Solution by H. B. LEONARD, B. S.

From $ax^2+by^2+cz^2=0=aax+b\beta y+c\gamma z$, $aa^2+\beta^2 b+c\lambda^2=1$, we get

$$x^2(aa^2+c\lambda^2)a+2aab\beta xy+y^2(b\beta^2+c\gamma^2)b=0,$$

$$\frac{x}{y}=-\frac{aab\beta\pm\gamma\sqrt{(-abc)}}{a(1-b\beta^2)},\qquad \frac{x}{z}=-\frac{aac\lambda\pm\beta\sqrt{(-abc)}}{a(1-c\gamma^2)},$$

$$\frac{y}{z}=-\frac{b\beta c\gamma\pm a\sqrt{(-abc)}}{b(1-aa^2)}.$$

Assuming a, b, c, a, β, γ to be real, then in order that $x:y:z$ may be real, $\sqrt{(-abc)}$ must be real. From $ax^2+by^2+cz^2=0$, it is clear that a, b, c can not all have the same sign and hence we must have one of the quantities a, b, c negative and the other two positive.

184. Proposed by J. A. CALDERHEAD, B.Sc., Professor of Mathematics, Curry University, Pittsburg, Pa.

If m rows, viz., the h_1th, h_2th, ...; h_mth, be transferred so as to become the 1st, 2nd, ..., mth, without altering the relative positions of the remaining rows, and that n columns, viz., the k_1th, k_2th, ..., k_nth, be similarly transformed the determinant thus obtained is the same as the original or differs from it only in sign according as $h_1+h_2+\ldots+h_m-\frac{1}{2}m(m+1)+k_1+k_2+\ldots+k_n-\frac{1}{2}n(n+1)$ is odd or even. [*Muir.*]

Solution by G. W. GREENWOOD, B. A. (Oxon), G. B. M. ZERR, A. M., Ph. D., and H. B. LEONARD, B. S.

In transferring the pth row (or column) to the qth row (or column) there are $p-q$ interchanges of adjacent rows (or columns) and therefore $p-q$ changes of sign. Hence, in the given example, there are

$$(h_1-1)+(h_2-2)+\ldots+(h_m-m)+(k_1-1)+(k_2-2)+\ldots+(k_n-n),$$
$$i.\ e.,\ h_1+h_2+\ldots+h_m-\tfrac{1}{2}m(m+1)+k_1+k_2+\ldots+k_n-\tfrac{1}{2}n(n+1)$$

changes of sign, and the determinant is unaltered in value, or differs only in sign, according as this value is *even* or *odd*; not odd or even as stated.

185. Proposed by L. E. DICKSON, Ph. D., Assistant Professor of Mathematics, The University of Chicago.

Without introducing radicals, eliminate x and y from the equations (1) $ax^2+bx+c=0$, (2) $ay^2+by+d=0$, and (3) $ax^2y^2+bxy+e=0$.

I. Solution by H. F. MacNEISH, A. B., Instructor in Mathematics, University High School, Chicago, Ill., G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va., and J. E. SAUNDERS, Hackney, Ohio.

The eliminant of (1) and (3) is

$$\begin{vmatrix} a, & b, & c, & 0 \\ 0, & a, & b, & c \\ ay^2, & by, & e, & 0 \\ 0, & ay^2, & by, & e \end{vmatrix}=0$$

or for $a \neq 0$,

$$(4) \qquad ac^2y^4 - b^2cy^3 + y^2(b^2c - 2ace + b^2e) - b^2ey + ae^2 = 0.$$

The eliminant of (2) and (4) is

$$\begin{vmatrix} ac^2 & -b^2c & b^2c-2ace+b^2e & -b^2e & ae^2 & 0 \\ 0 & ac^2 & -b^2c & b^2c-2ace+b^2e & -b^2e & ae^2 \\ a & b & d & 0 & 0 & 0 \\ 0 & a & b & d & 0 & 0 \\ 0 & 0 & a & b & d & 0 \\ 0 & 0 & 0 & a & b & d \end{vmatrix} = 0,$$

which reduces to

$$\begin{vmatrix} a(c^2d - b^2c + 2ace - b^2e) - b^2c(b+c), & b(abc - bcd - c^2d), & -a^2e^2 \\ a & b & d \\ bc(ac+bd), & ac^2d - (b^2cd - 2acde + b^2de - a^2e^2), & be(ae+bd) \end{vmatrix} = 0.$$

II. Solution by the PROPOSER.

To avoid the introduction of determinants of high order, we proceed thus: Multiply the third equation by a and replace $a^2x^2y^2$ by $ax^2 . ay^2$ obtained from the first and second.

$\therefore b(ax + bx + c)y + bdx + ae + cd = 0.$

Substituting in the second equation the value of y thus rationally determined, and dropping the factor a (the case $a=0$ being trivial), we obtain a second quadratic for x:

$$(a \mp b + d)b^2dx^2 + (2ade + 2cd^2 + bcd - abe - b^2e)bx + (ae + cd)^2 - b^2ce = 0.$$

The eliminant may now be determined in simple form.

186. Proposed by L. E. DICKSON, Ph. D., Assistant Professor of Mathematics, The University of Chicago.

Eliminate x and y from the equations (1) $ax^3 + bx^2 + cx + d = 0$, (2) $ay^3 + by^2 + cy + e = 0$, (3) $ax^3y^3 + bx^2y^2 + cxy + f = 0$, the eliminant to be rational in d, e, f.

Solved by H. F. MacNEISH, A. B., Instructor in University High School, Chicago, Ill., and G. B. M. ZERR A. M., Ph. D., Parsons, W. Va.

Using the same method as in No. 185.

GEOMETRY.

203. Additional solutions of problem 203 have been received from *G. W. GREENWOOD*, B.A. (Oxon) Professor of Mathematics and Astronomy, McKendree College, Lebanon, Ill., and *J. CHARLES RATHBUN*, A. B., Assistant in Physics, University of Washington.

205. Solutions of problem 205 have also been received from *G. B. M. ZERR*, A.M., Ph.D., Parsons, W. Va., and *G. W. GREENWOOD*, B. A. (Oxon) Professor of Mathematics and Astronomy, McKendree College, Lebanon, Ill.

206. Proposed by W. J. GREENSTREET, A. M., Editor of The Mathematical Gazette, Stroud, England.

$ABCD$ is circumscribed by a circle center O, and it circumscribes a circle radius r. The perpendiculars from C on the sides are x, y, z, u. Show that $\frac{1}{2}AC.BD=r\Sigma x$.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

The problem should read "the perpendiculars from O" instead of "the perpendiculars from C.

Let a, b, c, d denote the sides AB, BC, CD, DA, respectively; x the perpendicular on a; y, on b; z, on c; u, on d; R=circum-radius.

Then $x=\sqrt{(R^2-\frac{1}{4}a^2)}$. Now $R=AC/2\sin B$.

$AC^2=a^2+b^2-2ab\cos B=c^2+d^2+2cd\cos B$.

$$\therefore x=\frac{1}{2\sin B}\sqrt{(AC^2-a^2\sin^2 B)}=\frac{b-a\cos B}{2\sin B},\quad y=\frac{a-b\cos B}{2\sin B},$$

$$z=\frac{d+c\cos B}{2\sin B},\quad u=\frac{c+d\cos B}{2\sin B}.\quad \therefore r\Sigma x=\frac{r(a+b+c+d)-r(a+b-c-d)\cos B}{2\sin B},$$

$$r=\frac{2\sqrt{(abcd)}}{a+b+c+d},\quad a+c=b+d,\quad \cos B=\frac{a^2+b^2-c^2-d^2}{2(ab+cd)}.$$

$$\therefore \cos B=\frac{(a-c)(a+c)+(b-d)(b-d)}{2(ab+cd)}=\frac{(a+c)(a+b-c-d)}{2(ab+cd)}.$$

$$\therefore r\Sigma x=\frac{\sqrt{(abcd)}}{\sin B}-\frac{\sqrt{(abcd)}.(a+b-c-d)^2}{4(ab+cd)\sin B},\quad \sin B=\frac{2\sqrt{(abcd)}}{ab+cd},$$

$$4(ab+cd)\sin B=8\sqrt{(abcd)}.$$

$$\therefore r\Sigma x=\frac{ab+cd}{2}-\frac{(a+b-c-d)^2}{8}.\ \text{But } d=a+c-d.$$

$$\therefore r\Sigma x=\frac{ab+c(a+c-b)}{2}-\frac{(b-c)^2}{2}=\frac{ab+ac+bc-b^2}{2}$$

$$=\frac{ac+b(a+c)-b^2}{2}=\frac{ac+bd}{2}=\frac{1}{2}AC.BD.$$

207. Proposed by W. W. HART, University High School, Chicago, Ill.

According to Gauss the circumference of a circle can be divided into n equal parts by ruler and compass only, when n is a prime of the form $2^{2^p}+1$.

The following construction gives good partial results for n equals *any* integer. If AB is the diameter of the circle, and C is the vertex of the equilateral triangle ABC, and if D is a point on AB at the distance $2AB/n$ from A, then draw the line CD cutting the circle at E and F; E being the more remote from

C. $AE=1/n$ circumference approximately. For low values of n this method is very practical; is it practical in general? How great is the error?

I. Solution by H. F. MacNEISH, A. B., Instructor in Mathematics, University High School, Chicago, Ill.

Join OE. Let $\angle ACD=x$ and $\angle AOE=y$; then $DCB=60°-x$; $ADC=120°-x$; $DAE=90°-\frac{1}{2}y$; $AED=30°-x+\frac{1}{2}y$. $AD=2AB/n=4r/n$; $AB=AC=BC=2r$; $AO=OE=r$.

In $\triangle ADC$: $\dfrac{\sin x}{\sin(120-x)}=\dfrac{AD}{AC}=\dfrac{4r/n}{2r}=\dfrac{2}{n}$.

$\therefore \frac{1}{2}n\sin x=\sin(120-x)=\frac{1}{2}\sqrt{3}\cos x+\frac{1}{2}\sin x$.....(1). $(n-1)\sin x=\sqrt{3}\sqrt{(1-\sin^2 x)}$.

$\therefore \sin x=\sqrt{\dfrac{3}{n^2-2n+4}}$.....(2). $\therefore \sin(120-x)=\dfrac{n}{2}\sqrt{\dfrac{3}{n^2-2n+4}}$.....(3),

and $\cos(120-x)=\dfrac{n-4}{2\sqrt{(n^2-2n+4)}}$.....(4).

In $\triangle AOE$: $\dfrac{AE}{OE}=\dfrac{\sin y}{\sin(90-\frac{1}{2}y)}=\dfrac{\sin y}{\cos\frac{1}{2}y}$. $\therefore AE=\dfrac{r\sin y}{\cos\frac{1}{2}y}$.....(5).

In $\triangle AEC$: $\dfrac{AE}{AC}=\dfrac{\sin x}{\sin(30-x+\frac{1}{2}y)}$. $\therefore AE=\dfrac{2r\sin x}{\sin(30-x+\frac{1}{2}y)}$.....(6).

From (5) and (6), $\dfrac{\sin y}{\cos\frac{1}{2}y}=\dfrac{2\sin x}{\sin(30-x+\frac{1}{2}y)}$.

$\therefore \sin y[\sin(30-x)\cos\tfrac{1}{2}y+\sin\tfrac{1}{2}y\cos(30-x)]=2\sin x\cos\tfrac{1}{2}y$,

or $\sin y[-\cos(120-x)\cos\tfrac{1}{2}y+\sin\tfrac{1}{2}y\sin(120-x)]=2\sin x\cos\tfrac{1}{2}y$.

Hence from (1), $\sin y[-\cos(120-x)\cos\tfrac{1}{2}y+\frac{1}{2}n\sin\tfrac{1}{2}y\sin x]=2\sin x\cos\tfrac{1}{2}y$

or $\sin x[n\sin y\sin\tfrac{1}{2}y-4\cos\tfrac{1}{2}y]=2\sin y\cos\tfrac{1}{2}y\cos(120-x)$

or since $\sin y=2\sin\frac{1}{2}y\cos\frac{1}{2}y$, $\sin x[n\sin^2\frac{1}{2}y-2]=\sin y\cos(120-x)$.

Then from (2) and (4),

$$\frac{\sqrt{3}}{\sqrt{(n^2-2n+4)}}[n\sin^2\tfrac{1}{2}y-2]=\frac{n-4}{2\sqrt{(n^2-2n+4)}}\sin y.$$

$\therefore 2\sqrt{3}\left[\dfrac{n(1-\cos y)}{2}-2\right]=(n-4)\sin y$. $\therefore 3(n-n\cos y-4)^2=(1-\cos^2 y)(n-4)^2$.

$$\therefore \cos y=\frac{(n-4)[3n\pm\sqrt{(n^2+16n-32)}]}{4(n^2-2n+4)}.$$

Then for the positive value of the radical we obtain the following values of y for $n=3, 4, 5, \ldots$

n	$\cos y$	$\log \cos y$	y	$2\pi/n$	Error	Rate of error
3	$-\frac{1}{2}$	——	120°	120°	0	.0000
4	0	——	90°	90°	0	.0000
5	——	9.49107	71° 57′ 12″	72°	2′ 48″	.0007
6	$\frac{1}{2}$	——	60°	60°	0	.0000
7	——	9.79393	51° 31′ 23″	51° 25′ 43″	5′ 40″	.0018
8	——	9.84806	45° 11′ 14″	45°	11′ 14″	.0042
9	——	9.88248	40° 16′ 38″	40°	16′ 38″	.0069
10	——	9.90599	36° 21′ 18″	36°	21′ 18″	.0099
11	——	9.92286	33° 8′ 53″	32° 43′ 38″	25′ 15″	.0129
12	——	9.93545	30° 28′ 15″	30°	28′ 15″	.0157
24	——	9.98363	15° 38′	15°	38′	.0422
48	——	9.99516	8° 32′ 30″	7° 30′	1° 2′ 20″	.1389
90	——	9.99818	5° 14′ 30″	4°	1° 14′ 30″	.3104
180	——	9.99967	2° 15′	2°	15′	.1250
360	——	9.99995	49′ 40″ to 54′ 40″	1°	7′ 30″	.1250

The construction therefore has an error of over $1\frac{1}{2}\%$ for values of $n>12$, and for large values of n the error is very great.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, West Va.

Let O be the center of the circle, $AO=R=EO$. Then $CO=R\sqrt{3}$, $DO=R-4R/n=(R/n)(n-4)$. $\therefore$ tangent $DCO=(n-4)/n\sqrt{3}$.

$$\sin DCO=\frac{n-4}{\sqrt{(4n^2-8n+16)}}, \quad \cos DCO=\frac{n\sqrt{3}}{\sqrt{(4n^2-8n+16)}},$$

$$\sin CED=\frac{CO\sin DCO}{EO}=\frac{(n-4)\sqrt{3}}{\sqrt{(4n^2-84+16)}}.$$

$$\sin(90°+DOE)=\sin(180°-DEO-DCO)=\sin(DEO+DCO).$$

$$\therefore \cos DOE=\sin(DEO+DCO)=\frac{(n-4)[3n+\sqrt{(n^2+16n-32)}]}{4n^2-8n+16}.$$

For $n=3$, 4 and 6 the error is nothing.
For $n=5$ the side and angle are a trifle small.
For $n>6$ the side and angle are too large but the error varies.
For $n=8$, $\cos DOE=.70479$, $DOE=45°\ 11'\ 14.5''$, an error of $11'\ 14.5''$.
For $n=12$, $\cos DOE=.86186$, $DOE=30°\ 28'\ 25.7''$, an error of $28'\ 25.7''$.*
For $n=20$, $\cos DOE=.95091$, $DOE=18°\ 2'\ 40''$, an error of only $2'\ 40''$.

*In solution I, Mr. MacNeish finds the error for $n=12$ to be 28′15″, otherwise the two solutions agree. Ed.

For $n=72$, $\cos DOE=.99559$, $DOE=5^\circ\ 29'$, an error of $29'$.

For large values of n the error is much too great for any purpose.

Also solved by *J. E. SANDERS*, Hackney, Ohio.

CALCULUS.

165. Proposed by CAPT. T. C. DICKSON, Ordnance Department, United States Army, Washington, D. C.

Solve by integration the differential equation

$$\frac{d^2\xi}{dt^2}+\frac{A}{B}\left(\frac{d\xi}{dt}\right)^2-\frac{C}{B}=0,$$

in which A, B, C are given functions of ξ, but independent of t.

Solution by L. E. DICKSON, Ph. D., Assistant Professor of Mathematics, The University of Chicago.

In view of the nature of the coefficients, we regard ξ as the independent variable and t the dependent, the formulae of transformation being

$$\frac{d\xi}{dt}=1\div\frac{dt}{d\xi},\qquad \frac{d^2\xi}{dt^2}=-\frac{d^2t}{d\xi^2}\div\left(\frac{dt}{d\xi}\right)^3.$$

The given equation thus becomes

$$\frac{d^2t}{d\xi^2}-\frac{A}{B}\frac{dt}{d\xi}+\frac{C}{B}\left(\frac{dt}{d\xi}\right)^3=0.$$

Set $dt/d\xi=y$, whence $t=\int y\,d\xi$. Then $\dfrac{dy}{d\xi}-\dfrac{A}{B}y+\dfrac{C}{B}y^3=0$.

Divide by y^3 and set $z=y^{-2}$. The resulting differential equation

$$\frac{dz}{d\xi}+\frac{2A}{B}z-\frac{2C}{B}=0$$

is linear. By the usual method, we get

$$z=2e^{-\lambda}\left(\int\frac{C}{B}e^{\lambda}\,d\xi+k\right),\qquad \lambda\equiv 2\int\frac{A}{B}d\xi.\qquad \therefore\ t=\int z^{-\frac{1}{2}}d\xi.$$

169. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

Find the value of y from the Eulerian equation

$$y=\int\frac{dx}{(x+\sqrt{3})\sqrt[3]{(x^2+1)}}.$$

Solution by G. B. M. ZERR. A. M., Ph. D., Parsons, W. Va.

Let $x=\frac{\sqrt{3}(z+1)}{z-1}$, $z^3-1=u^3$.

$$\therefore \int \frac{dx}{(x+\sqrt{3})\sqrt[3]{(x^2+1)}} = -\int \frac{dz}{z\sqrt[3]{[4(z^3-1)}} = -\int \frac{udu}{\sqrt[3]{4(u^3+1)}}.$$

$$\therefore y=\frac{1}{3\sqrt[3]{4}}\int \frac{du}{1+u} - \frac{1}{6\sqrt[3]{4}}\int \frac{(2u-1)du}{1-u+u^2} - \frac{1}{2\sqrt[3]{4}}\int \frac{du}{1-u+u^2}.$$

$$\therefore y=\frac{1}{6\sqrt[3]{4}}\log\left(\frac{(1+u)^2}{1-u+u^2}\right) - \frac{1}{\sqrt{3}\sqrt[3]{4}}\tan^{-1}\left(\frac{2u-1}{\sqrt{3}}\right),$$

where $u=\frac{\sqrt[3]{[6\sqrt{3}(x^2+1)]}}{x-\sqrt{3}}$.

170. Proposed by W. J. GREENSTREET, A. M., Editor of The Mathematical Gazette, Stroud, England.

Find the center-locus of conics having 4-point contact with a given conic at a given point. Show that the conic of minimum eccentricity is given by $e^4\tan^2\varphi+4e^2-4=0$, where e is its eccentricity, and φ is the angle which the linear center-locus above makes with the normal to the curve at the point.

Solution by WILLIAM HOOVER, Ph. D., Professor of Mathematics in the State University, Athens, Ohio.

The coördinates of the center of any conic

$$ax^2+2hxy+by^2+2gx+2fy+c=0\ldots\ldots(1), \text{ are}$$

$$x_1=\frac{hf-bg}{ab-h^2}\ldots\ldots(2),\quad y_1=\frac{gh-af}{ab-h^2}\ldots\ldots(3),$$

and the eccentricity is given by

$$e^4+\frac{(a-b)^2+4h^2}{ab-h^2}(e^2-1)=0\ldots\ldots(4).$$

If the tangent and normal to the given curve be the axes of abscissas and ordinates, the equation of the conic having 4-point contact with the given conic is of the form

$$ax^2+2hxy+by^2+2gx-\lambda x^2=0\ldots\ldots(5), \text{ or, } (a-\lambda)x^2+2hxy+by^2+2gx=0\ldots\ldots(6).$$

Comparing (1) and (6), $a=a-\lambda$, $f=0$, $c=0\ldots\ldots(7)$, and (2) and (3) become

$$x_1=-\frac{bg}{(a-\lambda)b-h^2}\ldots\ldots(8),\quad y_1=\frac{gh}{(a-\lambda)b-h^2}\ldots\ldots(9).$$

(9)÷(8) gives $y_1-(h/b)x_1\ldots\ldots(10)$, the locus-center. Thus the "slope" is

$-(h/b)$, and the tangent of the angle (10) makes with the y axis is $\tan\theta = -(b/h)$.....(11). Substituting $a = a - \lambda$ in (4), we find

$$u = \frac{1}{e^2} = \frac{a-\lambda+b}{\sqrt{[(a-\lambda-b)^2+4h^2}} + \tfrac{1}{2}.....(12).$$

Equating $du/d\lambda$ to zero, we find $(a-\lambda)b - b^2 - 2h^2 = 0$, or,

$$a - \lambda = \frac{b^2+2h^2}{b}.....(13). \quad \text{Then } a - \lambda - b = \frac{2h^2}{b}.....(14).$$

Substituting (7), (13) and (14) in (4) and reducing

$$e^4 + \frac{4h^2}{b^2}(e^2-1) = 0, \text{ or } \frac{b^2}{h^2}e^4 + 4e^2 - 4 = 0.....(15).$$

This becomes by (11), $e^4\tan^2\varphi + 4e^2 - 4 = 0$.....(16).

Also solved by *G. W. GREENWOOD*, B. A. (Oxon), Professor of Mathematics and Astronomy in McKendree College, Lebanon, Ill., and by *G. B. M. ZERR*, A. M., Ph. D., Parsons, W. Va.

171. Proposed by J. E. SANDERS, Hackney, Ohio.

A thread passes spirally around a *rough* cylinder 10 feet high and 6 inches in diameter. How far will a pigeon fly in unwinding the thread if the distance between the coils is 4 inches, and the thread *unwound* is at all times *horizontal*?

I. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, West Va.

Let r=radius of cylinder=3 inches=$\frac{1}{4}$ feet; d=distance between coils =$\frac{1}{3}$ feet; m=number of coils=30. Then from Vol. I, No. 6, pages 318-320 of this Journal* we have for the required distance

$$S = \pi m^2\sqrt{(4\pi^2r^2+d^2)} \text{ feet} = 900\pi\sqrt{(\tfrac{1}{4}\pi^2 + \tfrac{1}{9})} \text{ feet} = 4540.246 \text{ feet, nearly.}$$

II. Solution by H. B. LEONARD, B. S.

Circumference$=6\pi$. Length of thread on one turn$=\sqrt{[(6\pi^2)+4^2]}$. Number of turns$=(12\times 10)\div 4=30$. Total rotation$=60\pi$. Angle of elevation of thread on cylinder$=\alpha=\sin^{-1}\frac{4}{\sqrt{[(6\pi^2)+4^2]}}$. During the bird's flight, an unwinding of $d\theta$ produces an increase of $dz=3\tan\alpha.d\theta$ in altitude, of $dr=3\sec\alpha.d\theta$ in direction of flight, of $dc=3\sqrt{(\theta^2\sec^2\alpha+1)}d\theta$ normal to direction of flight.

$$(ds)^2=(dr)^2+(dz)^2+(dc)^2=(9\sec^2\alpha+9\tan^2\alpha+9\theta^2\sec^2\alpha+9)(d\theta)^2$$
$$=(18\sec^2\alpha+9\theta^2\sec^2\alpha)(d\theta)^2. \quad ds=3\sec\alpha\sqrt{(2+\theta^2)}d\theta.$$

$$S=\int_0^{60\pi} 3\sec\alpha\sqrt{(2+\theta^2)}d\theta = 3\sec\alpha\int_0^{60\pi}\sqrt{(2+\theta^2)}d\theta = 3\sec\alpha\left[\tfrac{1}{2}\{\theta\sqrt{(2+\theta^2)}\right.$$

* See also Vol. 1, pp. 88-89. Ed.

$$+2\log[\theta+\sqrt{(2+\theta^2)}]\}\Big]_{\theta=0}^{\theta=60}=3\sec\alpha\{\tfrac{1}{2}[60\pi\sqrt{(2+3600\pi^2)}$$

$$+2\log[60\pi+\sqrt{(2+3600\pi^2)}]-\tfrac{1}{2}[0\sqrt{2}+2\log(0+\sqrt{2})]\}$$

$$=3\sec\alpha\{30\pi\sqrt{(2+3600\pi^2)}+\log[60\pi+\sqrt{(2+3600\pi^2)}]-\log\sqrt{2}\}=4540 \text{ feet.}$$

DIOPHANTINE ANALYSIS.

116. Proposed by HARRY S. VANDIVER, Bala, Pa.

If n is an odd positive integer, and 1, n, n', n'', ... denote all its distinct divisors, then $2^n>2[n+1][n'+1][n''+1]$.....

Solution by L. E. DICKSON, Ph. D., Assistant Professor of Mathematics, The University of Chicago.

There is a single exception $n=3$, for which $2^3=2[3+1]$. The corrected theorem may be proved by induction, using the following lemma:

If p is an odd prime number and d and π positive integers,

$$[1+d]^{p^\pi}>[1+d][1+pd][1+p^2d].....[1+p^\pi d],$$

except for $d=1$, $\pi=1$, $p=3$, the equality sign then holding.

For proof we apply $p^\pi-1=p-1+p[p-1]+p^2[p-1]+.....+p^{\pi-1}[p-1]$.

$$\therefore[1+d]^{p^\pi}=[1+d][1+d]^{p-1}[1+d]^{p(p-1)}.....[1+d]^{p^{\pi-1}(p-1)}.$$

But $[1+d]^{t(p-1)}\gtrless[1+td]^{p-1}\gtrless1+[p-1]td+[td]^{p-1}\gtrless1+[p-1]td+td$, if $p>2$. The equality signs hold simultaneously only when $t=d=1$, $p=3$. Hence, for $p>2$, $[1+d]^{t(p-1)}>1+ptd$ unless $t=d=1$, $p=3$, so that the lemma follows.

To prove the theorem by induction, we note that it is true for $n=p^\pi>3$, in view of the lemma for $d=1$. Assume that it has been verified for $n=p_1^{\pi_1}p_2^{\pi_2}.....$ We proceed to prove it true for $N=np^\pi$, p being prime to n. We have

$$2^N=[2^n]^{p^\pi}\gtrless[(1+1)(1+n)(1+n').....]^{p^\pi}$$

$$\gtrless[(1+1)(1+p).....(1+p^\pi)][(1+n)(1+pn)(1+p^2n).....][(1+n')(1+pn').....],$$

in view of the lemma. But the distinct divisors of N are

$$1, n, n', ..., p, pn, pn', ..., p^2, p^2n, ..., p^\pi, p^\pi n, ...$$

The theorem is therefore true for N.

118. Proposed by L. C. WALKER, A.M., Professor of Mathematics, Colorado School of Mines, Golden, Col.

Find the two least integral numbers such that their sum shall be a square and the sum of their squares a biquadrate.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let x and y be the numbers, then for $x+y=1$, $x^2+y^2=13^4$,

$x=120$, $y=-119$.

PROBLEMS FOR SOLUTION.

ALGEBRA.

189. Proposed by S. F. NORRIS, Professor of Astronomy and Mathematics, Baltimore City College, Baltimore, Md.

Solve $3x-y=xy-x^2$, $\sqrt{y}=\sqrt{\frac{2y-x}{2}}$.

GEOMETRY.

212. Proposed by B. F. FINKEL, A.M., M.Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Given two triangles ABC and $A'B'C'$ lying in the same plane. The side $B'C'$ cuts the sides AC, BC, and AB in the points I, H, and G, respectively; the side $A'B'$ cuts the same sides, AC, BC, and AB in D, F, and E, respectively; and $A'C'$ cuts AC, BC, and AB in M, L, and K, respectively. Prove that

$$(DA'.EA'.A'F)\ (GB'.HB'.B'I)\ (MC'.LC'.C'K)$$
$$=-(KA'.A'L.A'M)(FB'.B'E.B'D)\ (IC'.C'H.C'G).$$

213. Proposed by H. F. MacNEISH, A. B., Instructor in Mathematics, University High School, Chicago, Ill.

Construct an equilateral triangle which shall have its vertices in three given parallel lines.

214. Proposed by H. F. MacNEISH, A. B., Instructor in Mathematics, University High School, Chicago, Ill.

Inscribe in a given circle a triangle whose sides shall pass through three given points.

CALCULUS.

173. Proposed by J. E. SANDERS, Hackney, O.

Find the area of the greatest ellipse that can be inscribed in the quadrant of a given circle.

MECHANICS.

164. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

P balances W on a system of n movable pulleys of equal weight, each hanging by a separate string. If P is moved find the maximum acceleration of W.

AVERAGE AND PROBABILITY.

150. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

If the length of a circular arc be b and the radius vary uniformly, what is the aver- area of all the segments possible?

151. Proposed by L. E. DICKSON, Ph. D., Assistant Professor of Mathematics, The University of Chicago.

If A and B play tennis together, find the probability that A will win, given that a is the probability that A will win a given point and b the the probability that B will win the point.

NOTES.

The index for Vol. X will be mailed with the opening number for 1904.

Mr. C. Gilman has been appointed Instructor in Mathematics and Surveying at Harvard University.

Mr. Walter J. Risley has been appointed Instructor in Mathematics at the Armour Institute, Chicago.

Mr. C. R. Burger has been elected Assistant Professor of Mathematics at the Colorado School of Mines.

Dr. H. E. Hawkes has been promoted to an Assistant Professorship of Mathematics at Yale University.

Professor R. J. Aley, of the University of Indiana, has been elected to the editorship of the *Educator-Journal*.

Professor F. Anderegg, of Oberlin College, is away on leave of absence. He is spending the year in study and sightseeing in his native country, Switzerland.

Dr. Charles H. Ashton, previously Instructor in Mathematics at Harvard University, has been appointed Assistant Professor of Mathematics at the University of Kansas.

The Chicago Section of the American Mathematical Society will meet at St. Louis, Mo., during the Christmas recess.

The San Francisco Section of the American Mathematical Society met at the University of California on December 19, 1903.

The Central Association of Science and Mathematics Teachers held its annual meeting in Chicago on November 27 and 28, 1903.

An organization meeting of Teachers of Mathematics in ths Middle States and Maryland was held in affiliation with the Association of Colleges and Preparatory Schools of the Middle States and Maryland at Columbia University, on Saturday, November 28th, 1903, in Milbank Memorial Chapel, Teachers College.

Any of our subscribers wishing to correspond with Mr. Finkel on matters of business pertaining to the MONTHLY should send such correspondence directly to him at 204 St. Marks Square, Philadelphia, Pa. All subscriptions and requests for sample copies, etc., should be sent to W. C. Calland, Treasurer, Drury College, Springfield, Mo.

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ERRATA.

Page 78, for No. 106, read 112.
Page 115, next to last line, delete $-.0095M\sin^2\varphi$.
Page 117, last line, for Bulletin American Mathematical Society, read Science.
Page 243, ll. 15 and 24, and p. 244, l. 7, for $(-b\pm\sqrt{D})/a$, read $(-b\pm\sqrt{D})/2a$.
Page 243, lines 21, 25, 31, for $2(\mid a \mid +\sqrt{D})$, read $2 \mid a \mid +\sqrt{D}$.
Page 244, strike out 2 where it appears as a coefficient of $\sqrt{D}$.
Page 252, first line, read has p and only p solutions.

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[Prepared by Saul Epsteen.]

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